A Note on the Optimal Choice of the Monetary Policy Rule in the Lucas Business Cycle Model

by

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I. Introduction

In his important paper Robert Lucas (1972) developed an equilibrium business cycle theory which gave a microeconomic foundation for the statistical Phillips curve. There are both nominal and real disturbances in the economy and economic agents observe the price level which responds to these shocks and decide how much they wish to work. Because they can't judge correctly whether a price change is caused by a nominal or real disturbance, monetary policy is not neutral; the choice of the policy rule has a systematic influence on the agents decisions. However, classical invariance properties of money continue to hold: scale changes in money do not change equilibrium output at all. In the analysis of the choice of a monetary policy rule, Lucas used a very strong criterion and proved the Pareto-optimality of a fixed money supply rule. Thus, the result supports an intuitive proposition that decreased monetary uncertainty improves the allocative power of price signals and thereby makes people better off. In this paper we shall employ a weaker definition of optimality and find the best policy rule.

A slightly simplified version of the model of Lucas is presented in Section II. Section III compares two specific monetary policy rules and shows the nonoptimality of a constant money supply rule. In Section IV the social planning problem is con-
considered and it is shown that both a $k$-percent rule and a totally uninformative policy rule cannot solve the problem. The latter rule is then compared with a more general class of policy rules. Section V specifies a condition for an optimal policy rule and identifies a totally uninformative rule with an optimal rule. Conclusions appear in Section VI.

II. The Model

The economy in which we shall study the problem of optimal choice of monetary policy rules is a stationary version of the Samuelsonian overlapping generations economy. At the beginning of each period a variable size generation of identical individuals is born which lives for two periods, youth and old age. Consumption occurs only in old age, production takes place only in youth. Each member of the younger generation supplies $n$ units of labor which yield an equivalent amount of a homogeneous, perishable good for which he has no immediate use. When old, he spends all his cash holdings to obtain consumption. Money is the only store of value in the economy and existing balances are held exclusively by the old.

An individual's preference over consumption-work bundles is represented by the additive utility function

$$v(c', n) = u(c') - g(n)$$  \hspace{1cm} (1)

The function $u(\cdot)$ is increasing, twice continuously differentiable, strictly concave and $\lim_{c' \to 0} u'(c') = \infty$, $\lim_{c' \to \infty} u'(c') = 0$. The function $g(\cdot)$ is increasing and strictly convex. Here and in what follows, primed variables denote future values and unprimed variables denote current values. There are two stochastic elements in this economy. First, the size of the younger generation changes randomly from one period to the next. Let $\theta$ denote the size of the younger population. The random variable $\theta$ is non-negative and is drawn independently for each period from the stationary distribution. Thus there is only one market in the economy (this single market however could be interpreted as one of the two identical markets in the original Lucas model as well under an additional restriction on $\theta$). The second random variable is a transfer variable; at the beginning of the second period of life, cash balances are shrunk or augmented by a factor $x$. To eliminate from the model the wealth redistribu-
tional effects of monetary expansion it is assumed that the quantity of new money is proportional to the pretransfer balances. We assume, the continuous random variables $x$ and $x'$ are independent and are distributed in a known manner over the interval
The movement of the money supply is described by
\[ m' = mx, \]
where \( m \) is the money stock held by the old at the beginning of the current period. We assume that the pretransfer money supply is known, but the realized value of \( x \) is not known until next period so that confusion between nominal and real disturbances exists for some time. The young observe the current period price level which reflects both nominal and real shocks and they infer the value of \( \theta \) from this noisy price signal. Based upon their perceptions they decide how much they wish to work. Let \( p \) and \( p' \) be the current and future price level, respectively. Then the rate of return on labor is \( \frac{px'}{p'} - 1 \). Since the member of the older generation has a unit elastic demand for goods, his consumption is simply
\[ c' = n\left(\frac{px'}{p'}\right) \]
An individual's maximization problem is then
\[ \max_{0 \leq n \leq 1} E \left[ u\left(n\frac{px'}{p'}\right) - g(n) \right] p, \]
where \( n \) is the leisure endowment of the young. The first-order condition for an interior maximum is:
\[ E\left[\left(\frac{px'}{p'}\right)u'(n\frac{px'}{p'}) \right] p = g'(n) \quad (2a) \]
That is, it requires that the marginal disutility of work equal the expected marginal utility of future consumption, conditioned on current price.

Commodity market equilibrium requires that we find a price function \( p(m, x, \theta) \) such that the supply of goods by the young equals the real money supply by the old\(^1\).

Letting \( z = x/\theta \), equilibrium implies
\[ n = mz/p \quad (2b) \]
Since \( m \) is known to agents, changes in it will not have any influence on the equilibrium values of real variables; a doubling of the value of \( m \) will simply double the price level. Thus we specify a price function of the form
\[ p(m, x, \theta) = m\phi(z) \quad (3) \]
where \( \phi(\cdot) \) is a continuous, nonnegative function. Since agents know \( m \) and \( \phi(\cdot) \), by observing the price level they can observe the realized value of \( z \).

We define
\[ \phi(z) = z/\phi(x) \quad (4a) \]
\[ G(w) = wg'(w) \quad (4b) \]
\[ U(w) = wu'(w) \quad (4c) \]
The function \( G(w) \) is positive for all \( w > 0 \), and is clearly increasing. The function
$U(w)$ is increasing if future consumption and current leisure are gross substitutes, and is decreasing if they are gross complements. We suppose gross substitutability to give an econometric Phillips curve the appropriate slope. Assuming further that

$$G(0) = U(0) = 0; \ G(\bar{n}) > U(\bar{n}), \quad (5)$$

one rewrites the market equilibrium condition (2b) in the form

$$n = \phi(z) \quad (6a)$$

Then

$$px' / p' = m\phi(z)x' / [mz\phi(z')] = z'\phi(z) / [z\phi(z')] \quad (6b)$$

Substituting (6a) and (6b) and making use of (4b) and (4c), (2a) can be rewritten as

$$E[U(\theta' / \theta) \phi(z')] = G(\phi(z)) \quad (7)$$

The expectation on the LHS of (7) is taken over the unconditional probability distributions of $x'$ and $\theta'$, and over the distribution of $\theta$ conditional on $z$. The function $\phi(z)$ describes the working of the economy and equilibrium price is $mz / \phi(z)$.

**III. Is a $k$-percent Rule Really Optimal?**

In this section we consider whether eliminating noise in the process of money creation gives rise to gains or not. For this purpose, let us compare the two extreme cases; a $k$-percent rule under which price signals are clear, and a proportionate feedback rule $x_i = k\theta_i$ with $k$ being any positive number, under which price signals convey no information about $\theta$. Under the first rule equation (7) reduces to

$$E_{\theta'}[U(\theta' / \theta) \phi(\theta')] = G(\phi(\theta)) \quad (8)$$

Note that to denote output-per-capita, we use $\phi(\theta)$ instead of $\phi(1/\theta)$. Under the second rule $z$ is constant and a fixed output level, say $\phi_n$, solves the equation

$$E_{\theta, \theta'}[U(\theta' / \theta) \phi_n] = G(\phi_n) \quad (9)$$

In order to see how labor supply depends on the size of the younger population under a $k$-percent rule, differentiate (8) implicitly with respect to $\theta$ to obtain

$$d\phi(\theta) / d\theta = -E_{\theta'}[U'(\theta' / \theta) \phi(\theta')] / \theta'G'(\phi(\theta))$$

By virtue of the assumption that future consumption and current leisure are gross substitutes, $U' > 0$ so each member of the small population works longer to exploit the high price of his product.

As to the criterion to rank a perfectly informative policy rule relative to a totally uninformative rule, we propose to rank policy rules by the expected lifetime utility
of the young persons. Let \( E_c, E_s \) be the expected lifetime utility of the equilibrium allocation without and with signal noise, respectively. Then

\[
E_c = E_{c, \theta} \left\{ u \left[ (\theta'/u) \phi(\theta') \right] \right\} - E_s \left\{ g \left[ \phi(\theta) \right] \right\} 
= E_{s, \theta} \left\{ u \left[ (\theta'/u) \phi(n) \right] \right\} - g \left( \phi_n \right)
\]

(11a)

(11b)

Now given our criterion, Appendix I proves the following theorem:

**Theorem 1**

A proportionate feedback policy rule \( x_t = k\theta_t \) dominates in an expected utility sense a fixed money supply rule \( x_t = \theta \) with probability one.

Thus a 4% monetary growth rule proposed by Friedman (1948) is dominated by a monetary policy rule which reacts to the state of the economy. At the theoretical level the Lucas model cannot espouse the Friedman rule. We can interpret this paradoxical result in terms of the theory of the second best, first presented by Lipsey and Lancaster (1956), which suggests that piecemeal removal of imperfections does not always contribute to the welfare of the economy. Even if a \( k \)-percent rule allows price signals to convey perfect information about \( \theta \), as long as it cannot substitute for missing security markets, it will not improve welfare in the economy.

We have assumed \( U' > 0 \), but as long as a unique nontrivial solution to (8) and (9) exists, even if \( U' < 0 \) (consumption and leisure are gross complements) Theorem 1 still holds. Note that \( E_c = E_s \) if and only if \( U' = 0 \). In fact, if the utility function is of the form \( u = a \log \phi' + b, a > 0 \), the solution for \( \phi(\theta) \) and \( \phi_n \) are \( \phi(\theta) = \phi_n = G^{-1}(a) \) for all \( \theta \) and hence \( E_c \) equals \( E_s \).

Although we assumed that consumption takes place in old age only, it can be proved by following almost the same steps that the theorem still holds even if the young consume part of their output.

Polemarchakis and Weiss (1977) presented an example in which a \( k \)-percent rule is dominated by a "totally random" monetary policy. Interestingly enough for the problem at hand a "totally random" monetary policy and a proportionate feedback rule lead to the same resource allocation. Using our notation, their model can be described as follows. An individual chooses labor supply to maximize expected utility without referring to prevailing price, since fluctuations in the observed prices are attributed solely to monetary instability. Thus he maximizes the following function:

\[
E_{s, \theta} \left\{ U \left[ (\theta'/u) \phi(\theta') \right] \right\} - g \left( \phi_n \right)
\]

where \( n \) denotes labor supply. From the first-order condition, we have

\[
E_{s, \theta} \left\{ U \left[ (\theta'/u) \phi(\theta') \right] \right\} = G \left( \phi_n \right)
\]

Equilibrium in the commodity market implies that \( p = m/n \), so that \( p' = \theta' / \theta \). Substituting \( p'/p \) into the optimality condition, we obtain

\[
E_{s, \theta} \left\{ U \left( \theta' n / \theta \right) \right\} = G \left( n \right)
\]

This equation is exactly the same as equation (9). Then their
claim, based on specific utility function, that a volatile monetary policy dominates a k-percent rule can be regarded as an application of Theorem 1. In their model individuals ignore information which observed price conveys about \( \theta \) because of totally randomized monetary policy. On the other hand, under a proportionate feedback rule the price level is not contingent on \( \theta \), therefore people ignore price information. Although the two monetary policy rules are equivalent with respect to resource allocation, a proportionate feedback rule seems more realistic.

Now to see what is going on here, let us consider the variability of future consumption \( c' = \theta' \phi'(z')/\theta \), where \( \theta \) is fixed. Under a k-percent rule \( c' = \theta' \phi'(\theta')/\theta \) and under a proportionate feedback rule \( c' = \theta' \phi(\theta')/\theta \). The elasticity of consumption with respect to \( \theta' \) is unity in the noisy price case and is \( 1 + \theta' \phi'(\theta')/\phi(\theta') \) in the clear price case. If \( U' < 0 \), \( \phi'(\theta') > 0 \) and thus the elasticity exceeds one. A fixed money supply rule means more risky second period consumption and highly risk averse individuals suffer from unstable consumption. If \( U' > 0 \), the elasticity is positive but less than one. Future consumption is thus an increasing function of \( \theta' \) regardless of the curvature of the utility function. For a utility function which exhibits weak risk aversion, consumption is more variable in a noisy price model, but this is offset by increases in the level. In a noisy price case the supply of labor and hence the consumption of the old increases.

Before leaving this section, it must be emphasized that Theorem 1 ignores the difference in the amount of information required to conduct each monetary policy. To implement a k-percent rule, the monetary authority just keeps the stock of money constant through time. Under a feedback policy rule the authority needs to know the exact value of \( \theta_t \) in setting \( x_t \). If the real costs associated with the acquisition and processing of the information on \( \theta_t \) and in addition the costs of changing the money supply are not negligible, a feedback rule would not necessarily make everyone better off in a stationary equilibrium.

### IV. Efficiency of a Proportionate Feedback Rule

#### A. Social Planning Problem

As seen in the last section, the choice of a density function of the nominal transfer variable exerts an important influence on the expected lifetime utility associated with the equilibrium allocation. Before proceeding to identify the optimal density function of a monetary policy variable, we consider the centrally planned economy in
order to provide a yardstick by which the efficiency of competitive allocation can be judged.

A central planner commands the young generation to work \( n = \phi(\theta) \) and distributes the output equally among the old. His problem is to choose a labor supply function that maximizes the individual's expected utility

\[
E_{n, \phi} = \{u[(\theta'/\theta)\phi(\theta')]\} - E_{\phi}(\phi(\theta))
\]

subject to the continuous distribution function \( F(\theta) \) of the number of young persons. That is, the planner is supposed to maximize the "social welfare" criterion

\[
E_{p} = \int u[(\theta'/\theta)\phi(\theta')]dF(\theta)dF(\theta') - \int g[\phi(\theta')]dF(\theta)
\]

\[
= \int \{u[(\theta'/\theta)\phi(\theta')]dF(\theta)dF(\theta') - g[\phi(\theta')]dF(\theta')
\]

\[
= \int [u[(\theta'/\theta)\phi(\theta')]dF(\theta)\phi(\theta')]dF(\theta')
\]

where the integrals are taken over the relevant limits. Maximization of \( E_{p} \) requires that curly bracket to be maximized for all \( (\theta, \theta') \) over \( \phi \). Setting the first derivative equal to zero, we have

\[
\frac{\partial}{\partial \phi} \{u[(\theta'/\theta)\phi(\theta')]\} = \frac{\partial}{\partial \phi} \{u[(\theta'/\theta)\phi(\theta')]\} = g'[\phi(\theta')]
\]

Alternatively by multiplying both sides by \( \phi(\theta') \), we get

\[
\int U[(\theta'/\theta)\phi(\theta')]dF(\theta) = G[\phi(\theta')]
\]

Now by implicit differentiation

\[
\frac{d\phi(\theta')}{d\theta} = \frac{\int (1/\theta)U'[(\theta'/\theta)\phi(\theta')]dF(\theta)}{g''[\phi(\theta')] - \int (1/\theta^2)U''[(\theta'/\theta)\phi(\theta')]dF(\theta)}
\]

Since the denominator is positive

\[
\frac{d\phi(\theta')}{d\theta} \equiv 0 \Leftrightarrow U' \equiv 0
\]

Thus, comparing (13) with (10) we see that the slope of the labor supply function is reversed under the planning system unless \( U' = 0 \). This implies that a \( k \)-percent rule cannot solve the social planning problem. Also, since under a proportionate feedback rule labor supply is fixed, the competitive price system cannot achieve the solution to the planner's maximization problem. If \( U' = 0 \), labor supply is constant and is exactly the same as in market equilibrium.

We now compare the expected utility associated with the planning allocation \( E_{p} \) and the expected utility attainable under a linear feedback rule \( E_{n} \) by considering the difference in the utility. As mentioned just before, the market solution does not
satisfy the condition (12), so that one might well expect that \( E_p \) exceeds \( E_a \). Appendix II gives a formal proof that \( E_p \geq E_a \). This result together with Theorem 1 implies that \( E_p \geq E_a \geq E_c \).

B. Optimality of a Proportionate Feedback Rule

Thus far our welfare comparison has been restricted to the two rules of thumb, a proportionate feedback rule and a constant money supply rule. We now extend the set of monetary policy rules to be compared with a linear feedback rule and consider both stochastic feedback rules and stochastic nonfeedback rules. Under stochastic feedback rules the random variables \( \theta \) and \( x \) are not independently distributed. Under nonfeedback rules they are statistically independent. Let \( E_x \) denote the expected lifetime utility associated with the general case where both \( \theta \) and \( x \) fluctuate randomly. It is defined as

\[
E_x = E_{\theta, x} \mathbb{E}[\theta'(\theta)\phi(x')] - E_{\theta, x} \mathbb{E}[\phi(x)]
\]

By carrying out manipulations analogous to those involved in the proof of Theorem 1 in Appendix I, we finally reach

\[
\phi_x(E_x - E_a) \leq E_x \mathbb{E}_{\theta, x} \text{Cov}\{\phi(x'/\theta'), U[(\theta'/\theta)\phi_x]\}
\]

To determine the sign of the covariance in (15), one has to know the sign of \( \phi'(\cdot) \). Following Lucas, suppose that the joint density function of \( \theta \) and \( x \), \( \xi(\theta, x) \), satisfies the restriction that, for any fixed \( \bar{\theta} \), \( P[\theta \leq \bar{\theta} | x/\theta = z] \) is an increasing function of \( z \). Letting \( \eta(\theta | z) \) be the conditional density function of \( \theta \), the above probability is

\[
P(\bar{\theta}, z) = \int_0^z \eta(\theta | z) d\theta.
\]

The restriction imposed can be expressed as

\[
F_2(\theta, z) > 0 \quad \text{for all } (\theta, z)
\]

Let

\[
\mu(\theta) = \int \int U[(\theta'/\theta)\phi(x'/\theta')] \xi(\theta', x') d\theta' dx'
\]

Then

\[
\mu'(\theta) = -(1/\theta) \int \int \theta' \phi(x'/\theta') U'[(\theta'/\theta)\phi(x'/\theta')] \xi(\theta', x') d\theta' dx'
\]

Hence

\[
\mu'(\theta) \equiv 0 \iff U' \equiv 0
\]

Let \( H(z) \) denote the left-hand side of (7). Using \( \mu(\theta) \), it may be written

\[
H(z) = \int \mu(\theta) \eta(\theta | z) d\theta
\]

Then integrating by parts,

\[
H(z) = \mu(\bar{\theta}) - \int \mu'(\theta) F(\theta, z) d\theta,
\]
where $\beta$ is the upper limit of the range of $\theta$. Then we have

$$H'(z) = -\int_0^\beta \mu'(\theta) F z(\theta, z) d\theta$$

Now by implicit differentiation of $H(z) = G[\phi(z)]$,

$$\frac{d\phi(z)}{dz} = \frac{H'(z)}{G'[\phi(z)]}$$

Taking account of transitivity of $\iff$, we have

$$\frac{d\phi(z)}{dz} \iff U' \iff 0$$ (17)

It is easy to check that a relation in (17) is consistent with the property of the labor supply function implied by (10) for a $k$-percent rule. Lucas (1972, Theorem 4) has proved that $0 < \phi'(z) < 1$ under more stringent restrictions on the density functions of $\theta$ and $x$. Since it is enough for our purpose to determine only the sign of $\phi'(z)$, the second restriction which serves to make elasticity less than one was not imposed. From (17), we have $\phi'U' > 0$ as long as $U' \neq 0$. It then follows that

$$E_{\theta|x', s'} \text{Cov} \{\phi(z'/\theta'), U[(\theta'/\theta)\phi_s] \} \leq 0$$

for all $(\theta, x')$. To sum up:

**Proposition 1**

Suppose the function $F(\theta, z)$ satisfies the restriction (16) and $U' \neq 0$. Then resultant equilibrium is inefficient in the expected utility sense relative to the allocation under a proportionate feedback rule.

Under a $k$-percent rule the conditional distribution function of $\theta$ becomes a "step" function and shifts to the left as $z$ increases, so that, loosely speaking, Theorem 1 can be regarded as a special case of this proposition.

V. The Optimal Monetary Policy

A. Special Case

In the preceding section we found that a proportionate feedback policy rule dominates some other rules in addition to a $k$-percent rule. In this section instead of comparing the two specific policy rules, we directly examine the optimal monetary policy rule itself. It will be seen that a linear feedback rule satisfies the optimality condition to be specified below. Before proceeding to the discussions of the optimal policy rule, let us generalize the notion "noisy price signals". Up to this point only a proportionate rule $x_t = k0_t$ yields completely noisy price signals. In fact, however, certain stochastic feedback rules thoroughly cloud prices and induce the same re-
source allocation. If $\theta$ and $z$ are independently distributed, observation of $z$ does not convey information about $\theta$, so that (7) reduces to (9). Thus in the generalized noisy price signals case labor supply does not depend on $z^0$.

In this subsection agents are assumed to be risk neutral for second period consumption; $u = c'$. Nevertheless the marginal disutility from working is increasing; $g = (1/2)m^2$. Then it follows from (7) that

$$E_{\theta',z'}[\theta' \phi(z')] E(\theta' \mid z) = [\phi(z)]^2, \forall z$$

Let us define

$$[M(z)]^2 = E\left(\frac{1}{\theta} \mid z\right)$$

Then the above equation can be written as

$$\phi(z) = kM(z),$$

where $k = E_{\theta,c}[\theta M(z)]$. Using $k$, the expected utility we must maximize can be expressed as:

$$E_u = E_{\theta,c} \left[ \frac{1}{2} \phi(z)^2 \right] = E_{\theta,c} \left[ (1/\theta)E_{\theta',z'} \left[ \theta' \phi(z') \right] - (1/2) [\phi(z)]^2 \right] = E_{\theta,c} \left[ (1/\theta - (1/2) [M(z)]^2) k^2 \right] = (1/2) k^2 E_{\theta}$$

Consequently, maximizing $E_u$ is equivalent to maximizing $k$. Note that in the noisy price messages case $k_n = E_\theta [E(1/\theta)]^{1/2}$, while in the constant money supply case $k_c = E\theta^{1/2}$. By applying Jensen's inequality, $k_n > k_c$ can be proved, so that Theorem 1 holds for the utility function assumed here.

Suppose that the joint density function of $\theta$ and $x$ is such that the resultant joint densities of $\theta$ and $z$ are positive on the rectangular region $a \leq \theta \leq b$, $a \leq z \leq \beta$, $0 \leq a < b$, $0 \leq \alpha < \beta$ and are zero elsewhere. Then the maximand can be expressed as

$$k = \int_a^b \int_a^\beta \theta M(z) f(\theta, z) dz d\theta,$$

where $f(\theta, z)$ is the joint density function. Writing the joint density as $f(\theta, z) = f(z) f(\theta \mid z)$, the integral can be rewritten as

$$\int_a^\beta f(z) \int_a^\beta \theta M(z) f(\theta \mid z) d\theta dz = \int_a^\beta f(z) h(z) dz$$

which we must maximize subject to the integral constraint

$$\int_a^\beta f(z) [M(z)]^2 dz = E\theta^{-1}$$

This is a simple isoperimetric problem in the calculus of variations. To solve the problem, introduce the Lagrange multiplier $\lambda$ and define the functional:
The Euler equation is
\[ \frac{\partial}{\partial M} \{ h(z) + \lambda [M(z)]^2 \} = 0 \]
for all \( z \). Using the definition of \( h(z) \), the Euler equation means
\[ \frac{1}{2M(z)} \int_a^b \theta f(\theta | z) d\theta = -\lambda \]
Since this must hold for all \( z \), differentiating each side with respect to \( z \), we have
\[ \frac{1}{2M(z)} \int_a^b \theta f(\theta | z) \frac{\partial}{\partial z} \frac{\partial f(\theta | z)}{\partial \theta} d\theta - \frac{1}{2} \int_a^b \theta f(\theta | z) M'(z) d\theta = 0 \]
This can be rewritten as
\[ \int_a^b \theta \frac{\partial f(\theta | z)}{\partial z} d\theta = \frac{M'(z)}{M(z)} \int_a^b \theta f(\theta | z) d\theta \]
Applying integration by parts to the right-hand side integral, we have
\[ \frac{M'(z)}{M(z)} = \int_a^b \frac{\partial F(\theta | z)}{\partial z} d\theta - \int_a^b \theta \frac{\partial F(\theta | z)}{\partial \theta} d\theta \]  
(18)
Using \( \nu(\theta | z) \) as above to denote the density of \( \theta \) conditioned on \( z \) and integrating by parts, \([M(z)]^2\) can be written as
\[ [M(z)]^2 = b^{-1} + \int_a^b \theta^2 \nu(\theta | z) d\theta, \]
where \( F(\theta | z) = \int_a^\theta \nu(\theta | z) d\theta \). Differentiating both sides with respect to \( z \) gives
\[ 2M(z)M'(z) = \int_a^b \theta^2 \frac{\partial F(\theta | z)}{\partial z} d\theta \]  
(19)
Since \( M(z) > 0 \), (19) implies that \( M'(z) \) has the same sign as \( \frac{\partial F(\theta | z)}{\partial z} \). Then it follows that if \( \frac{\partial F(\theta | z)}{\partial z} \not= 0 \), condition (18) cannot be satisfied. Thus a monetary policy rule under which \( P[\theta \leq \bar{\theta} | x/\theta = z] \) depends on \( z \) cannot be an optimal policy rule. Now in the generalized noisy price signals case \( M(z) = (E\theta^{-1})^{1/2} \) and by assumption \( \frac{\partial F(\theta | z)}{\partial z} \)  
= 0, so that the underlying monetary policy rule does satisfy equation (18). Therefore, we have established that among monetary policy rules satisfying the conditions imposed above, a proportionate feedback rule is the best rule.

B. General Case
We next consider whether the above conclusion can be extended to the case where the utility function exhibits risk aversion. The preceding proof exploited special convenient properties of the preference function and thus cannot be applied for a general case. However, as we shall see, proceeding analogously as above, one finds
that a proportionate feedback rule still dominates any other rules. We continue to assume that the joint densities of \( \theta \) and \( z \) are positive only in the rectangular region; \( a \leq \theta \leq b, \ a \leq z \leq \beta \).

Our aim is to find a monetary policy rule that maximizes the expected lifetime utility

\[
\sum_{\theta} \int_{a}^{b} \int_{a}^{b} \left\{ u(\theta', \theta) \phi(\theta') - g(\phi(\theta')) \right\} f(\theta, \theta', z') \, d\theta \, d\theta' \, dz'
\]

subject to the constraint

\[
E\{U(\theta', \theta)\phi(\theta') | z\} = G(\phi(z))
\]

From (20a) it follows that

\[
E\{U(\theta', \theta)\phi(\theta')\} = E\{G(\phi(z))\}
\]

The restriction (20b) is weaker than the restriction (20a), so that if some monetary policy rule maximizes expected utility subject to (20b), and if it satisfies (20a), then it is a solution to the relevant problem stated above.

By the definition of the conditional probability function, \( f(\theta, \theta', z') = f(z') \, f(\theta, \theta' | z') \).

Then the expected utility can be expressed as

\[
\sum_{\theta} \int_{a}^{b} \int_{a}^{b} \left\{ u(\theta', \theta) \phi(\theta') - g(\phi(\theta')) \right\} f(\theta, \theta', z') \, d\theta \, d\theta' \, dz'
\]

The integral constraint can be rewritten as

\[
\int_{a}^{b} f(z') \left\{ G[\phi(z')] - s(z') \right\} \, dz' = 0
\]

Thus, one must maximize (21) subject to (22). Once again the problem is reduced to solving the variational problem. Every solution to this problem satisfies the Euler-Lagrange condition

\[
\frac{\partial}{\partial \phi} \{q(z') + \lambda [G(\phi(z')) - s(z')]\} = 0,
\]

where \( \lambda \) is a nonnegative multiplier independent of \( z' \). Using the definitions of \( q(z') \) and \( s(z') \), the Euler equation implies

\[
-\lambda = \sum_{\theta} \int_{a}^{b} \left\{ (\theta'/\theta) u(\theta'/\theta) \phi(\theta') - g(\phi(\theta')) \right\} f(\theta, \theta' | z') \, d\theta \, d\theta'
\]

This must hold for all \( z' \). Let us consider whether a proportionate feedback rule satisfies the Euler equation. Under this rule
\[ \int \sum_{n=1}^{\infty} (\theta' / \theta) u(\theta', \theta', \theta', z') f(\theta, \theta', z') d\theta d\theta' = g'(\phi_n), \]

so that \( \lambda = 0 \). Since \( \lambda \) is nonnegative and independent of \( z' \), a completely noisy monetary policy rule solves the above variational problem. Furthermore, by construction \( \phi_n \) satisfies the condition (20a), so that a linear feedback rule is the optimal monetary policy rule we are seeking.

Proposition 1 compares resource allocation under a proportionate feedback rule to allocations which result from other policy rules under which \( x \) is, roughly speaking, negatively correlated with \( \theta \). The above argument has established that even if \( x \) and \( \theta \) are positively correlated, an underlying monetary policy rule cannot dominate a linear feedback rule. It bears repeating that our use of the term "feedback rule" is different from the common use of the term. If the monetary authority makes the nominal transfer variable feedback upon some variables which are supposed to be included in the people's information set, for example, \( x_t = \omega(\theta_{t-1}, \theta_{k-1}) \), then the feedback rule is neutral; the resulting resource allocation is the same as under a \( k \)-percent rule.

VI. Conclusions

We analyze a slightly modified version of the Lucas model, reaching the conclusion that a proportionate feedback rule \( x_t = k\theta_t \) is best (on the basis of maximum lifetime utility of the young) in a variety of cases. Therefore the Lucas model offers little theoretical support to the view that a constant money supply is optimal once an economically rational welfare criterion is used. This seems to be an interesting and intuitively appealing result; but someone would say that the feedback strategy violates the rules of this genre of papers in that the monetary authority must have better information than the utility maximizers, since it must know \( \theta_t \), not just \( \theta_{t-1} \), in order to set \( x_t \). Yet, in fact the monetary authority undoubtedly believes that it reacts to exogenous shocks at least slightly faster than the rest of the economy does. Taking \( \theta \) as a proxy for the unforeseen part of general exogenous shocks, the demonstration that a feedback rule based on privileged information is optimal may be of practical significance.

Appendix I

Expanding \( \phi(\theta) \) and \( \phi(\theta') \) about \( \phi_n \) and neglecting terms of order higher than first, we obtain
Subtracting $E_n$ from both sides and defining $\Delta = E_n - E_n$, we have

$$
\Delta \leq E_{\theta'\theta'} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - E_{\theta} (\theta) - \phi_n \right\} - E_{\theta} (\theta) - \phi_n g' (\phi_n) + E_{\theta} (\psi') - E_{\theta} (\psi_n) + G (\phi_n)
$$

$$
= E_{\theta'\theta'} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - g' (\phi_n) - E_{\theta} (\theta) - \phi_n U \left( \frac{\theta'}{\theta} \phi_n \right) + G (\phi_n)
$$

$$
= E_{\theta'\theta'} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - g' (\phi_n) \right\}
$$

where $E_{\theta'\theta}$ is a conditional expectation operator. Multiplying both sides of the expression by $\phi_n$, we get

$$
\phi_n \Delta \leq E_{\theta'\theta'} \left\{ U \left( \frac{\theta'}{\theta} \phi_n \right) - G (\phi_n) \right\}
$$

$$
= E_{\theta} \left\{ U \left( \frac{\theta'}{\theta} \phi_n \right) - G (\phi_n) \right\} + E_{\theta} (\psi') - E_{\theta} (\psi_n) + E_{\theta} (\psi') - E_{\theta} (\psi_n) + E_{\theta} (\psi') - E_{\theta} (\psi_n) + G (\phi_n)
$$

$$
= E_{\theta} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - g' (\phi_n) \right\}
$$

Here we used the fact that the expected value of the product of two random variables is equal to the product of their expected values plus their covariance. As seen in the text, $\psi (\theta') U \left( \frac{\theta'}{\theta} \phi_n \right) < 0$, so that covariance between $\psi (\theta')$ and $U \left( \frac{\theta'}{\theta} \phi_n \right)$ is negative for any given $\theta$. Q. E. D.

Appendix II

Proceeding exactly as in the comparison of $E_\alpha$ and $E_n$ in Appendix I, we obtain

$$
E_n - E_\alpha \leq E_{\theta'\theta'} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - E_{\theta} (\psi) - g' (\phi_n) \right\}
$$

$$
= E_{\theta'\theta'} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - E_{\theta} (\psi) - g' (\phi_n) \right\}
$$

$$
= E_{\theta'\theta'} \left\{ (\theta' - \phi_n) u' \left( \frac{\theta'}{\theta} \phi_n \right) - E_{\theta} (\psi) - g' (\phi_n) \right\}
$$

$$
= \phi_n E_{\theta'\theta'} \left\{ U \left( \frac{\theta'}{\theta} \phi_n \right) - E_{\theta} (\psi) - g' (\phi_n) \right\}
$$

$$
= 0
$$

Q. E. D.
FOOTNOTES

1) After writing this note, Lucas (1983) proposes to modify the definition of the equilibrium price function as Grondmont has pointed out a logical error in his original paper (1972). Modified as outlined in his corrigendum, the model still allows us to compare various monetary policy rules. Throughout this note, we follow Lucas and employ a narrower definition of equilibrium.

2) Note that our criterion is the same as an equal treatment Pareto-optimality (ET-PO) proposed by Muench (1977).

3) Azariadis (1981) shows the same result for a utility function \( v(c', n) = c' - (1/2)n^2 \).

4) See Lucas (1980) for a recent restatement of Friedman's proposal.

5) By Theorem 3 in Lucas (1972), 
\[ 0 < \left[ \frac{(1/\theta')\phi'(1/\theta')}{\phi(1/\theta')} \right] < 1, \text{ where } \phi(1/\theta') = \frac{1}{\theta' \phi'(\theta')} \].

6) To take an example, let 
\[ f(\theta, x) = \frac{2}{(a^2)(k_2-k_1)} \]
be the joint density, and assume that 
\[ 0 \leq \theta \leq a, \ k_1 \theta \leq x \leq k_2 \theta, \ a > 0, k_2 > k_1 > 0 \].
The joint probability density of \( \theta \) and \( z \) is given by 
\[ f(\theta, z) = \frac{2}{(a^2)(k_2-k_1)} \]
if \( 0 \leq \theta \leq a, \ k_1 \leq z \leq k_2, = 0 \) elsewhere. Consequently, \( \theta \) and \( z \) are statistically independent.

BIBLIOGRAPHY


