A Two-Stage Model of Credibility

by

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Abstract

This paper presents a simple two-stage model of credibility which analyzes situations that arise when someone, who must decide whether to trust his partner or not, is uncertain whether irresponsible action is deliberate or accidental. If the initial credit placed on the partner is considerably high relative to the probability of making mistakes, transaction continues for two periods regardless of the outcome. Otherwise, a sequential equilibrium involves mixed strategies played by the both players. The effects of changes in parameter values on the equilibrium types and strategies as well as on expected payoffs to the players are also analyzed.

1. Introduction

Credibility plays a vital role in the efficient operation of any economy. First rate department stores must select dependable suppliers to maintain their reputation among customers; banks need to trust loan borrowers to provide funds; and employers delegate responsibility to their employees in accordance with their reliability. Agents gain trust through their reliable actions without which they may suffer a loss of customers or business contracts. However, even responsible people sometimes make mistakes like inadvertently bouncing a check. This paper attempts to model situations that arise when someone, who must decide whether to trust people he deals with, is uncertain about whether irresponsible action is deliberate or accidental.

There are two types of agents: the Senders (S) and the Receiver (R) who are to play two periods. S contains two types of players: good types (G) who have similar

1) I am grateful to Joel Sobel whose work (Sobel (4)) inspired me to initiate this paper.
preferences with R, and bad types (B) who have opposed preferences to R. At the outset of the first period, R selects S randomly. After learning the outcome of S’s action, R decides whether to continue to play with S, or to replace S with another player whose credibility is the same as S’s original credibility. In order to do this, R must reassess the credibility of S. The difficulty arises because R is uncertain whether the bad outcome was a deliberate result on the part of B or an accidental result on the part of G.

The solution of this game employs the concept of sequential equilibrium introduced by Kreps and Wilson (2). The equilibrium has the following form. When the probability R places on S being good is large enough, replacement of S will not take place regardless of the outcome. Put differently, S’s high credit will not be impaired completely by one failure. Taking advantage of this, B uses a pure strategy of duping R. Otherwise, there is a positive probability that even G will be replaced, and the equilibrium will involve mixed strategies used by both B and R. The lower the original reputation, the more likely B will take responsible action which is not optimal to B in the short run in order to enhance his reputation.

In what follows, I will restrict my attention to an example of a department store (R) and suppliers (S) for ease of exposition. Section 2 sets up a model, and Section 3 analyzes the model and presents the main results. Section 4 concludes.

2. The Model

A fixed population of suppliers is composed of a fraction P1 of good types (G) and (1-P1) of bad types (B). These types are unknown to R. At the start of period one, R selects a supplier randomly, so the probability that R places on S being G is P1. S moves first, supplying either high quality product (h) or low quality product (l). Payoff to the suppliers on the product type supplied are given in Table 1. In the short run, it is optimal for G to supply h and B to supply l. The values a and b should lie in the following range.

0<a, b<1.

Observing the quality of the product, R chooses either to continue to buy from the same S or to replace him with a new supplier. For the replacement, R must pay c, and

<table>
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<th>Table 1. Quality of Product</th>
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<td>Types of S</td>
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the probability of the new supplier being G is assumed to be \( P_1 \). R makes either 1 or \(-1\) depending on whether the product turns out to be \( h \) or \( l \). These payoffs to R embody the idea that the negotiation cost required for R to make S compensate for \( l \) is too costly. The game continues for two periods, and the second period's payoffs are discounted with a discount factor \( r \).

G may supply \( l \) by accident. To model this I assume that there are two moves \( h^* \) and \( l^* \) available to S, and that \( h \) is a joint consequence of both the action \( h^* \) and nature's move. Specifically, when S plays \( h^* \), the product will be \( h \) with probability \( q \) and \( l \) with probability \( 1-q \). It is also assumed that \( l^* \) will lead to \( l \) with probability one as in Figure 1. For notational convenience the same symbol \( l \) is used to identify both the move \( l^* \) and the outcome \( l \). Based on the initial belief about S and the observed outcome, R revises \( P_1 \) to \( P_2 \). B and G receive the payoffs \( v \) and \( w \) from their alternative jobs respectively when dismissed.

A sequential equilibrium comprises a strategy for each player and a function \( P_2 \) such that: (a) Starting either from the first or the second period, S's strategy is the best response to R's strategy. (b) For each \( n=1, 2 \), R's strategy is the best response to S's strategy given that the supplier is good with probability \( P_n \). (c) \( P_2 \) is computed from \( P_1 \) and S's strategies and nature's move using Bayes' rule.

There is a tradeoff for type B supplier. Providing high quality product enhances his reputation, but only at the expense of foregoing immediate gain by supplying low quality product. G always plays \( h^* \) since playing \( l \) will not only yield the smaller short run gain but also hurt his reputation.

3. Existence, Uniqueness, and Properties of the Equilibrium

To begin analysis of this game, note that at the second period (the final stage), B must never play \( h^* \), since to do so lowers his payoff with no possible compensation.

R's Action at Period 2: Suppose that \( P_2 \) is the probability reassessed by R that S is a good supplier. \( h^* \) yields a payoff \( 2q-1 \) and \( l \) yields \(-1\) to R. Thus R will expect to receive \(- (1-P_2) + P_2 (2q-1) = 2qP_2 - 1\) from the original supplier. See Figure 2. There
are two other options open to R. First, R can switch the supplier to another S who has the same credibility as the original supplier by paying the replacement cost $c$. The new supplier will give the expected payoff $2qP_1 - 1 - c$ to R. Secondly, no transaction, which gives a value of zero, is also a possible choice to R.

Therefore, depending on the values of $P_1$ and $P_2$, R takes three different actions. If $2qP_1 - 1 - c$ is nonnegative and greater than $2qP_2 - 1$, then it is profitable for R to replace S with a new supplier. If $2qP_2 - 1 < 2qP_1 - 1 - c < 0$, then R will choose not to purchase from any producers. Refer to Figure 3. Let $P_2^*$ be the critical value of the credibility $P_2$ at which R is indifferent between whether to continue or discontinue with the original S. To sum up, $P_2^* = P_1 - c / 2q$ for $P_1 \geq (1 + c) / 2q$, and $P_2^* = 1 / 2q$ for $P_1 < (1 + c) / 2q$. I will take up the $P_1 \geq (1 + c) / 2q$ case first.

**Case 1.** $P_1 \geq (1 + c) / 2q$.

In this case, suppliers are expected to be sufficiently reliable, so R wishes to start transacting with a new supplier when the current S is dismissed.

S's Action at Period 1: As mentioned earlier, G chooses $h^*$. There are three strategies that B can take.

1. Pure Strategy $h^*$.
2. Pure Strategy $l$. 
3. Pure Strategy $t$. 

Figure 3.
(3) Mixed Strategy (Probability distribution over $h^*$ and $l$).

To make the problem interesting, it is assumed that the payoff to $B$ from the alternative job $v$ is strictly less than one. Let $P_i$ denote the probability that $B$ plays $i$, where $i=h^*$ or $l$. Observing the product quality, $R$ reevaluates the credibility of $S$. $P_i$ is updated as follows:

$P_i^h = \text{Prob}(S \text{ is } G | \text{quality } h)$

$$= \frac{\text{Prob}(S \text{ is } G \text{ and quality } h)}{\text{Prob}(\text{quality } h)}$$

$$= \frac{\text{Prob}(h | G) \text{Prob}(G)}{\text{Prob}(h | G) \text{Prob}(G) + \text{Prob}(h | B) \text{Prob}(B)}$$

(1) $$= \frac{qP_1}{qP_1 + (1-q)P_2} = \frac{P_1}{P_1 + (1-P_1)P_2},$$

and

$P_i^l = \text{Prob}(S \text{ is } G | \text{quality } l)$

$$= \frac{\text{Prob}(G \text{ and quality } l)}{\text{Prob}(l)}$$

$$= \frac{\text{Prob}(l | G) \text{Prob}(G)}{\text{Prob}(l | G) \text{Prob}(G) + \text{Prob}(l | B) \text{Prob}(B)}$$

(2) $$= \frac{(1-q)P_1}{(1-q)P_1 + (1-P_2)q + (1-q)P_2(1-P_1)},$$

Does $B$ always try to supply $h$ being afraid of losing $R$? In an equilibrium this does not happen.

Lemma 1: There is no pure strategy equilibrium with $P_h=1$.

Proof: Assume that a strategy pair $(P_h=1, P_2*)$ with the following Bayesian updating rule

$$P_i^h = qP_1/[qP_1 + q(1-P_1)] = P_1,$$

and

$$P_i^l = (1-q)P_1/[1-P_2] = P_1,$$

constitutes a Bayesian Nash Equilibrium. Since $P_i^h = P_i^l = P_1 > P_2^*$, $R$ will continue with $S$ regardless of the outcome. This means that $B$ will profitably deviate from $h^*$ to $l$ with no possibility of being dismissed. Q.E.D.

Next, consider a strategy pair $(P_h=0, P_2*)$. The Bayes' rule for this strategy requires that

$$P_i^h = qP_1 = 1 > P_2^*$$

and

$$P_i^l = \frac{(1-q)P_1}{1-qP_1} = \frac{(1-q)P_1}{1-qP_1}.$$

If $P_i^l$ exceeds $P_2^*$, then $R$ will choose to continue with the original $S$ even when $l$ is
observed. The condition that $P_1' \geq P_2'$ can be expressed in terms of $P_1$ for given $q$ and $c$ as

$$\frac{(1-q)P_1}{1-qP_1} \geq P_1 - \frac{c}{2q}$$

which simplifies to

$$2q^2P_1^2 - q(2q+c)P_1 + c \geq 0.$$  

Intuitively speaking, the above inequality implies that the bad outcome cannot shake R's confidence that S is a good supplier. R will continue to purchase from the S no matter what product's quality is. In this case B always plays $l$. In what follows, I assume that R decides to play the game. As will be shown later, this assumption is innocuous since the expected total payoff to R is always positive when $2qP_1$ is greater than one, which is true for the current case.

Lemma 2: If the condition (4) holds, there exists a sequential equilibrium in which B plays $l$ with probability one, and R continues to play with the original S no matter which outcome occurs.

Proof: Suppose R takes the above strategy. Playing the pure strategy $P_\alpha=0$ yields the total payoff $1+r$ to B since B can sell his products for both periods with probability one. By defecting and playing $h^*$, B gives up unit $b$ immediately with probability $1-q$, and gains $r$ in the second period. Clearly, $P_\alpha=0$ is the best response for B to the R's strategy.

From the way (4) is constructed, R responds optimally to the B's strategy $P_\alpha=0$ by staying with the original S regardless of the outcome. Q.E.D.

In this case, R strongly believes that the S facing R is a good supplier, so that R will regard the observed unfavorable outcome as being accidentally made by G. Thus, no penalty will be imposed on $l$. Taking advantage of this strong belief of R in S, B plays $l$ with probability one.

If (4) does not hold, then there is a positive probability that B plays $h^*$. If the bad supplier plays $l$, then he will gain one unit this period but only $v$ in the second period since R removes S judging the bad outcome $l$ to be sufficient evidence that he is B.

Lemma 3: Assume that (4) does not hold, and $v < 1-b/r$. Given that an equilibrium exists, B must play $h^*$ with a probability strictly between 0 and 1.

Proof: If B plays $l$, he will be unable to sell to R in the second period, so that his total payoff will be

$$1+rv.$$  

If B deviates from this strategy and plays $h^*$, then it will supply $h$ with probability
1−q. For the former outcome, B gains immediate payoff 1−b and makes one in the second period, while for the latter outcome, B makes one in period one and v in period two. Thus the present value of the total expected payoff to B by defecting to h* will be

\[(6)\ (1−q)(1+rv)+q(1−b+r)\].

It follows that (6) is greater than (5) if and only if

\[(7)\ v<1−b/r\].

Therefore, it will be in the bad supplier's interest to defect, implying that \(P_h=0\) is not an equilibrium strategy. Together with the result obtained above, one can conclude that the strategy used by B must be a mixed strategy if an equilibrium is to exist. Q.E.D.

Corollary: If \(v\geq1−b/r\), \(P_h=0\) is an equilibrium strategy.

The reason should be obvious. When the alternative job is sufficiently attractive to B, it is in his interest to always supply a low quality product.

Lemma 4: If \(v<1−b/r\), and (4) does not hold, the following is an equilibrium.

(A) B randomizes, playing \(l\) with the probability \(P_l\) given by

\[(8)\ P_l=1−P_h=c(1−q)/q(1−P_l)(2qP_l−c),\]

and \(h^*\) with the complementary probability.

(B) R replaces S with another S with probability \(b/r(1−v)\) if \(l\) is observed. R continues to play with S if \(h\) is observed.

Proof: Suppose that R takes the strategy described in (B). The expected payoff to B, when \(l\) is supplied, will be

\[\frac{b}{r(1−v)}(1+rv)+(1−\frac{b}{r(1−v)})(1+r),\]

which simplifies to \(1+r−b\).

When \(h\) is supplied, the payoff to B will be \(1−b+r\). Therefore, B is indifferent between strategies \(h^*\) and \(l\), so that the randomized strategy given in (A) is the best response to R's strategies (B).

Set \(P_l\) equal to \(P_l^*=P_l−c/2q\), that is,

\[(9)\ P_l^*=\frac{P_l(1−q)}{P_l(1−q)+(1−P_l)(1−P_hq)}=P_l−\frac{c}{2q}.\]

This can be rewritten as

\[2qP_l(1−q)=(2qP_l−c)[P_l(1−q)+(1−P_l)(1−q+q−P_hq)].\]

Rearranging terms yields

\[(10)\ 1−P_h=\frac{c(1−q)}{q(1−P_l)(2qP_l−c)},\]

which is consistent with the S's strategy described by (A). This implies that, as far as B plays (A), R is indifferent whether to continue with the S or to replace him. Thus, the
R's strategies in (B) is the best response to (A). Q.E.D.

The following proposition summarizes the results so far obtained. For derivation of the parameter restrictions, refer to Appendix 1.

Proposition 1: For the $P_1 > (1+c)/2q$ case, depending on parameter values, there are the following types of equilibria.

1. If given $v > 1 - b/r$, then a strategy pair $(P_0, P_1^*)$, and beliefs $P_0^* = 1$, and $P_1^* = P_1(1-q)/(1-qP_1)$ will constitute a sequential equilibrium. B uses the pure strategy $I$.

   The strategy of R depends on the values of $P_1$. If $P_1$ is no less than $P_1^*(q, c) = [(2q+c) + \sqrt{(2q+c)^2 - 8c}]/4q$, then $P_2^* > P_2^*$ so that R continues with the S regardless of the outcome. Otherwise, R replaces S with a new supplier when the bad outcome is observed.

2. Suppose $v < 1 - b/r$. Then, there are two subcases:
   
   (i) $P_1 > P_1^*(q, c)$. There is an equilibrium characterized by a strategy pair $(P_0 = 0, P_2^*)$ and beliefs $P_0^* = 1$, $P_1^* = P_1(1-q)/(1-qP_1) > P_2^*$. R continues to deal with S no matter what outcome occurs. B, taking advantage of this, always supplies $I$.

   (ii) For $P_1 < P_1(q, c)$, the strategies and beliefs given by (1), (2), and (A), (B) in Lemma 4 constitute a sequential equilibrium.

Remarks: 1. For $v \geq 1 - b/r$, B has no incentive to play $h^*$ and to secure the current customer at the cost of foregoing immediate gain because he can receive sufficiently high payoff from the alternative job.

2. The parameter restrictions given in 2(i) imply that R is strongly confident that S is G relative to the probability of S's making mistakes. Therefore, R judges the outcome $I$ more likely resulted from G's mistake than from B's intention. B must foresee that his choice of action cannot influence the future course of the game, so that he will maximize his immediate gain, which means playing $I$.

From (8), $P_0$ decreases as c increases. The larger the cost of replacement, the less likely R will replace S, and the more often B will behave irresponsibly. (A-14) implies that B acts more responsibly when the probability of G's making mistakes is smaller.

Next, consider the payoff to R, and S. When $P_0 = 0$, no replacement takes place, so that R, if faced with G, gains 1 with probability $q$ and loses 1 with probability $1-q$ for both periods. If played with B, R loses one in both periods. Thus, the expected total

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2) If $q < (3c+1)/(2(1+c))$, this condition is automatically satisfied.

3) To understand this point, note that the region for the pure strategy ($P_0 = 0$) equilibria is above the upward sloping curve $P_1^*$ as in Figure A-2.
payoff to R, when $P_1 \leq P_1^*$, will be

\[ E_r^R = P_1(2q-1)(1+r) - (1-P_1)(1+r) \]

\[ = (1+r)(2qP_1-1), \]

which is positive since $2qP_1 > 1 + c$.

For $P_h > 0$, R is indifferent whether to continue to transact with S, or to replace him when $l$ occurs. Suppose that R actually chose to play with the original S. In this case, the second period payoff to R is $-1$ when S is $G$, and $2q-1$ when S is $B$. The relevant payoffs and probabilities are given below.

\[
\begin{array}{c|c|c|c|c|}
 & h_+ & h_- & h^* & h^* \\
\hline
G & q & h (1) & l (-1) \\
1-q & h (1) & l (-1) \\
\hline
S & 1-q & h (1) & l (-1) \\
\hline
B & 1-q & l (-1) & l (-1) \\
1-p_h & l (-1) & l (-1) \\
\end{array}
\]

Figure 4.

Therefore, the expected payoff to R, for $P_1 < P_1^*$, will be

\[ E_r^R = P_1(2q-1)(1+r) + (1-P_h)(2q-1-r) + (1-P_h)(-1-r). \]

This simplifies to

\[ E_r^R = (1+r)(2qP_1-1) + 2qP_h(1-P_h). \]

Again, $E_r^R$ must be positive for the $2qP_1 > 1 + c$ case. R will always participate in the game with S. Substituting (8), (13) reduces to

\[ E_r^R = 2q(1+rP_1) + 1-r + 2q(c-2P_1)/(2qP_1-c). \]

The effect of an increase in $P_1$ on $E_r^R$ can be seen by differentiating (14) with respect to $P_1$.

\[ \frac{\partial E_r^R}{\partial P_1} = 2q + \frac{4cq(1-q)}{(2qP_1-c)^2} > 0. \]

Thus R will be better off when there are relatively more good suppliers in the economy although this tends to raise the probability of $P_1$.

Differentiating (14) with respect to $q$ gives

\[ \frac{\partial E_r^R}{\partial q} = 2(1+rP_1) + \frac{2c(2P_1-c)}{(2qP_1-c)^2} > 0, \]

which means that R will benefit from reduced probability of S's making mistakes. Refer to Figure 5.

4) A reader can verify that the two moves give the same expected payoff to R.
As already seen, B's payoff is quite simple. As shown in Figure 6, he earns $1+r$ if $P_1 \geq P_1^*$, and $1+r-b$ otherwise. Simple computation shows that $P_1^*(q, c)$ is a decreasing function of $c$. Thus, for $P_1 < P_1^*$, it is possible that an increase in $c$ causes B to gain an additional payoff $b$. It should be clear from Figure 6 that B prefers to have a small $q$ relative to $P_1$.

Let us investigate the expected payoff to G. For $P_1 \geq P_1^*$, G makes one with probability $q$ and $1-b$ with the complementary probability for both periods. Hence, for $P_1 \geq P_1^*$,

\[ E_{\pi G} = (1+r)(1-b+bq). \]

For $P_1 < P_1^*$, there is a positive probability $P_c$ that transaction is discontinued and G receives $w$ from the alternative employment provided that the bad outcome $l$ occurs. Thus, for $P_1 < P_1^*$,

\[ E_{\pi G} = q[1+r(q+(1-q)(1-b))]+(1-q)[1-b+P_crw+(1-P_c)r(q+(1-q)(1-b))]. \]

(13) reduces to

\[ E_{\pi G} = (1-b+bq)(1+r)-P_c r(1-q)(1-b+bq-w). \]

From inspection of (13), it follows that

\[ \frac{\partial E_{\pi G}}{\partial q} > 0. \]

To avoid trivial cases, assume that G prefers dealing with R to the alternative opportunity, that is, $1-b+bq > w$. Remembering $P_c = b/r(1-v)$, it is obvious that an increase in $v$ will lead to a reduction in G's payoff. The $v$-increase tends to make B act less responsibly, having a negative external effect on G through increased probability of replacement. $E_{\pi G}$ is depicted in the following figure.

**Case 2.** $P_1 < (1+c)/2q$:

In this case, it is not in R's interest to replace S with another supplier when S is judged unreliable since the prospective supplier is also trustless. R would rather not
engage in business. As noted earlier, the cutoff level of credibility equals $P_a^*=1/2q$. Therefore, $P_a$ is determined implicitly by

$$P_a(1-q) = 1 - P_a(1-q) + (1-P_a)(1-P_a^*)2q.$$

There are two types of equilibria as in the previous case. Attention will be restricted on the more interesting case $v<1-b/r$. All aspects of the game remain unaltered except for $P_a$ and the payoff to $R$ for $P_2$ revised below $P_2^*$.

**Proposition 2:** Suppose that $v<1-b/r$ and $P_1<(1+c)/2q$. If $2qP_1>1$, then there exists a sequential equilibrium. If, in addition, $2qP_1^2-3P_1^2q+1\geq0$, then the equilibrium will involve a pure strategy $P_a=0$ on the part of $B$. Otherwise, a mixed strategy equilibrium arises in which both $B$ and $R$ randomize with $P_a$ given by (21) and $P_c=b/r(1-v)$ respectively.

**Proof:** Omitted. See Appendix 2 for the parameter restrictions.

Consider, next, the payoffs to the players. For $P_1\geq P_1^*$, $E_R, E_B$, and $E_G$ are the same as those in the previous case since the transaction takes place for both periods no matter which outcome occurs. In particular, $R$ is willing to play the game as far as $2qP_1>1$. For $P_1<P_1^*$, the expected total payoffs to $S$ remain unchanged. However, the total expected payoff to $R$ differs from the former case. Given that the bad outcome occurs, the conditional expected payoff to $R$ becomes zero since $R$ is indifferent between dealing with the original $S$ and no transaction. Therefore,

$$E_R^R=P_1[2q-1+q(2q-1)r]+(1-P_1)(P_a(2q-1-qp)+(1-P_a)(-1)),$$

which can be reduced to

5) Again which type of equilibrium arises depends on parameter values. Appendix 2 derives the regions for pure and mixed strategy equilibria in $(P_a, q)$ plane.
En-R = 2qP_1 - 1 + rqP_1(2q - 1) + (1 - P_1)P_4q(2 - r).

Clearly, if 2qP_1 > 1, (23) is positive, implying that R will participate in this game.

Combining Figure A. 2 and A. 3, one can conclude that there are the following four regions in (P_1, q) plane:

Region I ...... Pure strategy equilibrium characterized by P_b = 0. B always plays l, taking advantage of high credibility that R places on S. The transaction continues for two periods with probability 1.

Region II ...... Mixed strategy equilibrium. If l occurs, there is a positive probability that S will be replaced.

Region III ...... Mixed strategy equilibrium. In the case of the bad outcome, there is a positive probability that R will transact with nobody in the second period.

Region IV ...... The game is not played.

As expected, Region II will shrink, and Region III will expand as c increases. Finally,

\[ P_1 = \frac{1 + c}{2} \]

\[ P_2 = 1/q(3 - 2q) \]

\[ P_3 = 2q + c + \sqrt{(2q + c)^2 - 8c} \]

Figure 8.

6) In Figure 8, loci for constant P_b's are also drawn by dotted lines. A locus in lower position corresponds to a higher P_b.
what if making commitment is possible? Then, R will continue with S if and only if \( h \) is supplied. It should be obvious that R prefers this, and both B and G will be worse off.

4. Conclusion

This paper analyzes a situation which arises when an agent is unsure about the motives of someone upon whom he must depend. In particular, the agent cannot tell whether a harmful outcome is an accidental result on the part of a reliable partner, or an intentional result on the part of an irresponsible partner. The extent to which the agent trusts the partner will be based on his earlier action which may differ from what he intended. Thus, there is an incentive for a bad partner to behave like a good partner in order to enhance gains from his future opportunity. Typically, there is a positive probability that even a good partner will be dismissed. However, if the agent holds strong confidence that he faces a good partner, one bad outcome will not destroy the credibility immediately. Taking advantage of this, a bad player will deceive the agent.

The assumption that the transaction continues only for two periods is artificial. It should be desirable to construct a long finite or an infinite horizon model. Such an extension may give rise to an interesting intertemporal behavior of a bad player like repeatedly building up and cashing in on his reputation.

Appendix 1

It follows from (8) and \( P_\alpha > (1+c)/2q \) that \( 1 - P_\alpha > 0 \), thus \( P_\alpha < 1 \). Consider the restriction \( P_\alpha \geq 0 \), that is,

\[
(A-1) \quad \frac{c(1-q)}{q(1-P_1)(2qP_1-c)} \leq 1.
\]

Rearranging terms yields

\[
(A-2) \quad 2qP_1^2 - q(2q+c)P_1 + c \leq 0.
\]

If the determinant is positive, there are two distinct real roots \( \bar{P}_1 \) and \( \bar{P}_2 \) (\( \bar{P}_1 < \bar{P}_2 \))

\[
(A-3) \quad \bar{P}_1, \bar{P}_2 = \frac{2q+c \pm \sqrt{2q+c}^2 - 8c}{4q}
\]

such that the solutions for (A-2) satisfy

\[
(A-4) \quad \bar{P}_1 < P_1 < \bar{P}_2.
\]

It is straightforward to show

\[
(A-5) \quad \bar{P}_1 < (1+c)/2q.
\]
In order for the determinant to be positive

\[(A-7) \quad q < \frac{(-c - \sqrt{8c})}{2}, \text{ or } q < \frac{(-c + \sqrt{8c})}{2}.\]

The former condition is irrelevant since \(q\) is positive.

To find the condition for \(\bar{P}_1 > \frac{(1+c)}{2q}\), multiply both sides by \(4q\), and simplify terms to get

\[-2q + c + 2 < \sqrt{(2q + c)^2 - 8c}.\]

This can be expressed as

\[(A-8) \quad q > \frac{3c + 1}{(c+1)},\]

It is also straightforward to show

\[(A-9) \quad -c + \frac{\sqrt{8c}}{2} < \frac{3c + 1}{2(c+1)}.\]

Rearranging terms gives

\[\frac{\sqrt{8c}}{2} < \frac{(c^2 + 4c + 1)}{(c+1)}.\]

Squaring both sides and simplifying yields

\[c^4 + 2c^2 + 1,\]

which is positive as desired.

Next, it needs to be shown that if \(q \geq \frac{(-c + \sqrt{8c})}{2}\), \(P_1\) cannot exceed \(\frac{(1+c)}{2q}\). To see this, note that, under the above restriction, \(\frac{(1+c)}{2q}\) takes on its minimum value \(\frac{(1+c)}{(-c + \sqrt{8c})}\) at \(q = \frac{(-c + \sqrt{8c})}{2}\). It is easy to see that this minimum value is no less than one. Therefore, there is no \(P_1 \leq 1\) that is larger than \(\frac{(1+c)}{2q}\).

The results so far obtained can be summarized as follows:

If \(q \geq \frac{(3c + 1)}{2(c+1)}\), then \(q \geq (\sqrt{8c})/2\) so that there is \(\bar{P}_1^*\) (henceforward denoted by \(P_1^*\)) strictly less than one such that

\[P_1 = 0 \text{ for } P_1 \geq P_1^*,\]

and

\[P_1 = 1 - c(1-q)/q(1-P_1)(2qP_1 - c) \text{ for } (1+c)/2q \leq P_1 < P_1^*.\]

If \(q < (3c + 1)/(c+1)\), then \(P_1 = 0\) for \(P_1 \geq (1+c)/2q\).

To depict the regions for the pure strategy equilibria and the mixed strategy equilibria in \((P_1, q)\) plane for given \(c\), denote \(qP_1\) by \(x\), and rewrite \((A-2)\) as follows:

\[(A-10) \quad q \geq x - \frac{c}{2} + \frac{c}{2x}.\]

Simple computation reveals that the right hand side of \((A-10)\) takes on its minimum value \(\sqrt{2c} - c/2\) when \(x = \sqrt{c}/2\). The corresponding minimum value of \(P_1\) is equal to
2/(2√2 − √c). See Figure A.1. From the inspection of (A−1), it can be seen that there are two \( P_1 \)'s satisfying (A−2) with equality for \( q \geq √2 c−c/2 \). Are there also two \( q \)'s satisfying (A−2) with equality for a given \( P_1 \)? Noting that \( P_1 \) stays constant when \( q \) moves on \( q = kx \), where \( k \) is some constant, the answer to this question is “Yes” if and only if the solutions to the following equation have two real roots \( x_1 \) and \( x_2 \) strictly between zero and one.

\[
(A-11) \quad x - \frac{c}{2} + \frac{c}{2x} = kx.
\]

The equation (A−11) reduces to

\[
(A-12) \quad 2(k-1)x^2 + cx - c = 0.
\]

Since the determinant is \( c^2 + 8c(k-1) \), there are two real roots if \( k > 1 - c/8 \). Assuming \( k \) exceeds \( 1 - c/8 \),

\[
(A-13) \quad x_1, x_2 = \frac{-c \pm \sqrt{c^2 + 8c(k-1)}}{4(k-1)}.
\]

If \( k > 1 \), \( x_1 < 0 \). \( x_2 \) is greater than one if

\[
\frac{-c + \sqrt{c^2 + 8c(k-1)}}{4(k-1)} > 1
\]

Resulting in the following series of inequalities.

\[
\sqrt{c^2 + 8c(k-1)} > 4(k-1) + c
\]

\[
c^2 + 8c(k-1) > 16(k-1)^2 + 8c(k-1) + c^2
\]

\[
0 > 16(k-1)^2,
\]

which is impossible. Therefore, for \( k > 1 \), \( x_1 < 0 \), and \( x_2 < 1 \). If \( k < 1 \), \( x_1 > 0 \), but \( x_2 > 1 \) from the above argument. Hence, for each \( P_1 \) there is at most one \( q \) which satisfies (A−2) with an equality. From (A−3), if follows that \( P_1 = c/2 \) if \( q = 1 \). Moreover, it is easy to see from (A−1) that for a given \( q \), \( P_1 \) is maximized when \( P_1 = 1/2 + 1/4q \).

Differentiating (8) with respect to \( q \),

\[
(A-14) \quad \frac{\partial P_1}{\partial q} = \frac{1}{(1-P_1)q} + \frac{2P_1}{(2qP_1-c)^2}(2q-c).
\]

(A−14) is positive since \( 2qP_1 > 1 + c \). Simple computation also yields

\[
(A-15) \quad q = x + \frac{c(1-x)}{2P_1 + (1-P_1)c}.
\]

Clearly, \( q \) approaches one as \( P_1 \) goes to zero.

These results together with (A−5) are summarized in Figure A.2.

Appendix 2

Rewrite (21) as

\[
(A-16) \quad 2qP_1(1-q) = P_1(1-q) + (1-P_1)(1-q+q-P_1q).
\]
Figure A.1

Figure A.2
This simplifies to

\[(A-17) \quad P_t = 1 - P_h = \frac{1 - q}{q} \cdot \frac{2qP_1 - 1}{1 - P_1}.\]

Since \(P_h\) must be strictly less than one in order for an equilibrium to exist, \(q\) needs to exceed \(1/2P_1\). The restriction that \(P_h\) is nonnegative is expressed as

\[(A-18) \quad (1 - q)(2qP_1 - 1) \leq q(1 - P_1),\]

which reduces to

\[(A-19) \quad 2P_1q^2 - 3P_1q + 1 \geq 0.\]

From \((A-17)\), \(P_1\) can be expressed in terms of \(P_t\) and \(q\):

\[(A-20) \quad P_1 = \frac{1/q - 1 + P_t}{2(1 - q) + P_t}.\]

Substituting \(P_t = 0\), 1 into \((A-20)\) gives

\[(A-21) \quad P_t = \frac{1}{q(3 - 2q)} \quad \text{for} \quad P_h = 0,\]

and

\[\text{Figure A.3}\]
Differentiating (A-18) with respect to $P_t$ yields

\[
\frac{dP_t}{dP_t} = \frac{(2q-1)(q-1)}{q},
\]

so that $P_t$ is increasing in $P_t$ when $q$ takes on values between $1/2$ and $1$, and decreasing in $P_t$ elsewhere. (A-19) attains its minimum value $8/9$ at $q=3/4$. $P_t$ equals $1$ at $q=1/2$ for any value of $P_A$. It also takes on $1$ at $q=1$ for every $P_A$ except when $P_A=1$. These results are shown in Figure A. 3.

REFERENCES


